Wattanawitoon et al. Journal of Inequalities and Applications 2012, 2012:138
http://www.journalofinequalitiesandapplications.com/content/2012/1/138

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A new hybrid general iterative algorithm for common solutions of generalized mixed equilibrium problems and variational inclusions

Kriengsak Wattanawitoon[1], Thanyarat Jitpeera[2] and Poom Kumam[2*]

* Correspondence: poom.kum@kmutt.ac.th
[2]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology (KMUTT), Bangmod, Thrungkru, Bangkok 10140, Thailand
Full list of author information is available at the end of the article

Abstract

In this article, we introduce a new general iterative method for finding a common element of the set of solutions generalized for mixed equilibrium problems, the set of solution for fixed point for nonexpansive mappings and the set of solutions for the variational inclusions for β_1, β_2-inverse-strongly monotone mappings in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above sets under some suitable conditions. Our results improve and extend the corresponding results of Marino and Xu, Su et al., Tan and Chang and some authors.

Mathematics subject Classification: 46C05; 47H09; 47H10.

Keywords: nonexpansive mapping, inverse-strongly monotone mapping, generalized mixed equilibrium problem, variational inclusion

1. Introduction

Let C be a closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$. Let F be a bifunction of $C \times C$ into $\mathcal{R}$, where $\mathcal{R}$ is the set of real numbers, $\Psi : C \to H$ be a mapping and $\varphi : C \to \mathcal{R}$ be a real-valued function. The *generalized mixed equilibrium problem* for finding $x \in C$ such that

$$F(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by GMEP(F, ϕ, Ψ), that is

$$\mathrm{GMEP}(F, \varphi, \Psi) = \{x \in C : F(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}.$$

If $F \equiv 0$, the problem (1.1) is reduced into the *mixed variational inequality of Browder type* [1] for finding $x \in C$ such that

$$\langle \Psi x, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by MVI(C, ϕ, Ψ).

If $\Psi \equiv 0$, the problem (1.1) is reduced into the *mixed equilibrium problem* for finding $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by MEP(F, ϕ).

If $\phi \equiv 0$, the problem (1.3) is reduced into the *equilibrium problem* [2] for finding $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by EP(F). See, e.g. [3-6] and the references therein.

If $F \equiv 0$ and $\phi \equiv 0$, the problem (1.1) is reduced into the *Hartmann-Stampacchia variational inequality* [7] for finding $x \in C$ such that

$$\langle \Psi x, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $VI(C, \Psi)$.

If $F \equiv 0$ and $\Psi \equiv 0$, the problem (1.1) is reduced into the *minimize problem* for finding $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by *Argmin*(ϕ).

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\theta(x) = \frac{1}{2}\langle Ax, x \rangle - \langle x, y \rangle, \quad \forall x \in F(S), \tag{1.7}$$

where A is a linear bounded operator, $F(S)$ is the fixed point set of a nonexpansive mapping S and y is a given point in H [8].

Recall, a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty [9]. A mapping $S : C \to C$ is said to be a *k-strictly pseudo-contraction* [10] if there exists $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C,$$

where I denotes the identity operator on C. A mapping A of C into H is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A mapping A of C into H is called an *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A mapping A of C into H is called *α-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

A linear bounded operator A is called *strongly positive* if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

A self mapping $f : C \to C$ is called *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. The *variational inclusion problem* is to find $x \in H$ such that

$$\theta \in B(x) + M(x), \tag{1.8}$$

where θ is the zero vector in H. The set of solutions of problem (1.8) is denoted by $I(B, M)$. The variational inclusion has been extensively studied in the literature. See, e.g. [11-15] and the reference therein.

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle > 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$.

Let B be an inverse-strongly monotone mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$, and define

$$Mv = \begin{cases} Bv + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then M is a maximal monotone and $\theta \in Mv$ if and only if $v \in VI(C, B)$ [16].

Let $M : H \to 2^H$ be a set-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \quad x \in H \tag{1.9}$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is the identity mapping. In the worth mentioning that the resolvent operator is nonexpansive, 1-inverse-strongly monotone and that a solution of problem (1.8) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$ (see also [17]).

In 2000, Moudafi [18] introduced the viscosity approximation method for nonexpansive mapping and proved that if H is a real Hilbert space, the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in C$ is chosen arbitrarily,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, \quad n \geq 0, \tag{1.10}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies certain conditions, converge strongly to a fixed point of S (say $\bar{x} \in C$) which is the unique solution of the following variational inequality.

In 2005, Iiduka and Takahashi [19] introduced following iterative process $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0, \tag{1.11}$$

where $u \in C$, $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\beta$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping (say $\bar{x} \in C$) which solve some variational inequality.

In 2006, Marino and Xu [8] introduced a general iterative method for nonexpansive mapping. They defined the sequence $\{x_n\}$ generated by the algorithm $x_0 \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n, \quad n \geq 0 \tag{1.12}$$

where $\{\alpha_n\} \subset (0, 1)$ and A is a strongly positive linear bounded operator. They proved that if $C = H$ then the sequence $\{x_n\}$ converges strongly to a fixed point of S (say $\bar{x} \in H$) which is the unique solution of the following variational inequality.

In 2008, Su et al. [20] introduced the following iterative scheme by the viscosity approximation method in a real Hilbert space: $x_1, u_n \in H$

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(u_n - \lambda_n Au_n), \end{cases} \tag{1.13}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy some appropriate conditions. Furthermore, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to the same point z, where $z = P_{F(S)\cap VI(C,A)\cap \mathrm{EP}(F)} f(z)$.

In 2011, Tan and Chang [14] introduced following iterative process for $\{T_n : C \to C\}$ be a sequence of nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(SP_C((1 - t_n)J_{M,\lambda}(I - \lambda A)T_n(I - \mu B))x_n), \quad \forall n \geq 0, \tag{1.14}$$

where $\{\alpha_n\} \subset (0, 1)$, $\lambda \in (0, 2\alpha]$ and $\mu \in (0, 2\beta]$. The sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of nonexpansive mapping, the set of solutions of the variational inequality and the generalized equilibrium problem.

In this article, we modify by Marino and Xu [8], Su et al. [20] and Tan and Chang [14], the purpose of this article, we show that under some control conditions the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of nonexpansive mappings, the solution of the generalized mixed equilibrium problems and the solution of the variational inclusions in a real Hilbert space.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. Recall that the metric (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| .$$

The following characterizes the projection P_C. We recall some lemmas which will be needed in the rest of this article.

Lemma 2.1. *The function $u \in C$ is a solution of the variational inequality* (1.5) *if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda\Psi u)$ for all $\lambda > 0$.*

Lemma 2.2. *For a given $z \in H$, $u \in C$, $u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0$, $\forall v \in C$.*

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0. \tag{2.2}$$

Lemma 2.3. [21]*Let $M : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a monotone and Lipshitz continuous mapping. Then the mapping $L = M + B : H \to 2^H$ is a maximal monotone mapping.*

Lemma 2.4. [22]*Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality* $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$, *hold for each $y \in H$ with $y \neq x$.*

Lemma 2.5. [23]*Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathcal{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$.

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [24]*Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then I - S is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, x_n - Sx_n \to 0$$

implies $x = Sx$.

For solving the mixed equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathcal{R}$ and $\varphi : C \to \mathcal{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each fixed $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(A4) for each fixed $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous;

(B1) for each $x \in C$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0,$$

(B2) C is a bounded set.

Lemma 2.7. [25]*Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $F : C \times C \to \mathcal{R}$ *be a bifunction mapping satisfies* (A1)-(A4) *and let* $\varphi : C \to \mathcal{R}$ *is convex and lower semicontinuous such that* $C \cap dom\varphi \neq \emptyset$. *Assume that either* (B1) *or* (B2) *holds. For* $r > 0$ *and* $x \in H$, *then there exists* $z \in C$ *such that*

$$F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $T_r^{(F,\varphi)} : H \to C$ *as follows:*

$$T_r^{(F,\varphi)}(x) = \left\{z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}$$

for all $x \in H$. *Then, the following hold:*

(i) $T_r^{(F,\varphi)}$ *is single-valued;*

(ii) $T_r^{(F,\varphi)}$ *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\left\|T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y\right\|^2 \leq \left\langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y\right\rangle;$$

(iii) $F(T_r^{(F,\varphi)}) = \mathrm{MEP}(F, \varphi)$;

(iv) MEP(F, ϕ) *is closed and convex.*

Lemma 2.8. [8]*Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \leq ||A||^{-1}$, *then* $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.9. [26]*Let H be a real Hilbert space and* $A : H \to H$ *a mapping.*

(i) *If A is a* δ*-strongly monotone and* μ*-strictly pseudo-contraction with* $\delta + \mu > 1$, *then I - A is a contraction with constant* $\sqrt{(1 - \delta)/\mu}$.

(ii) *If A is a* δ*-strongly monotone and* μ*-strictly pseudo-contraction with* $\delta + \mu > 1$, *then for any fixed number* $\tau \in (0, 1)$, $I - \tau A$ *is a contraction with constant* $1 - \tau(1 - \sqrt{(1 - \delta)/\mu})$.

3. Strong convergence theorems

In this section, we show a strong convergence theorem which solves the problem of finding a common element of $F(S)$, GMEP(F_1, ϕ_1, B_1), GMEP(F_2, ϕ_2, B_2), $I(A_1, M_1)$ and $I(A_2, M_2)$.

Theorem 3.1. *Let H be a real Hilbert space, C be a closed convex subset of H. Let* F_1, F_2 *be bifunctions of* $C \times C$ *into* $\mathcal{R}$ *satisfying* (A1)-(A4) *and* $A_1, A_2, B_1, B_2 : C \to H$ *be* $\beta_1, \beta_2, \eta, \rho$*-inverse-strongly monotone mappings,* $\varphi_1, \varphi_2 : C \to \mathcal{R}$ *be convex and lower semicontinuous functions,* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$, M_1, $M_2 : H \to 2^H$ *be maximal monotone mappings and A is a* δ*-strongly monotone and* μ*-strictly pseudo-contraction with* $\delta + \mu > 1$, γ *is a positive real number such that*

$\gamma < \frac{1}{\alpha}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)$. *Assume that either* (B1) *or* (B2) *holds. Let S be a nonexpansive mapping of C into itself such that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F_1, \varphi_1, B_1) \cap \mathrm{GMEP}(F_2, \varphi_2, B_2) \cap I(A_1, M_1) \cap I(A_2, M_2) \neq \emptyset.$$

Suppose $\{x_n\}$ *is a sequences generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n B_1 x_n) \\ v_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B_2 x_n) \\ x_{n+1} = \xi_n P_C \left\{\alpha_n \gamma f(x_n) + (I - \alpha_n A) S P_C [J_{M_1,\lambda_1}(1 - \lambda_1 A_1) u_n]\right\} \\ +(1-\xi_n) P_C [J_{M_2,\lambda_2}(I - \lambda_2 A_2) v_n], \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0, 1)$, $\lambda_1 \in (0, 2\beta_1)$ *such that* $0 < a_1 \leq \lambda_1 < b_1 < 2\beta_1$, $\lambda_2 \in (0, 2\beta_2)$ *such that* $0 < a_1 \leq \lambda_2 \leq b_2 < 2\beta_2$, $r_n \in (0, 2\eta)$ *with* $0 < c \leq d \leq 1 - \eta$ *and* $s_n \in (0, 2\rho)$ *with* $0 < e \leq f \leq 1 - \rho$ *satisfy the following conditions:*

(C1): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C2): $0 < \liminf_{n\to\infty} \xi_n < \limsup_{n\to\infty} \xi_n < 1$, $\sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty$,

(C3): $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(C4): $\liminf_{n\to\infty} s_n > 0$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta$, *where* $q = P_\Theta(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle(\gamma f - A)q, p - q\rangle \leq 0, \quad \forall p \in \Theta$$

which is the optimality condition for the minimization problem

$$\min_{q\in\Theta} \frac{1}{2}\langle Aq, q\rangle - h(q), \tag{3.2}$$

where h is a potential function for γf *(i.e.,* $h'(q) = \gamma f(q)$ *for* $q \in H$*).*

Proof. Since A_1, A_2 are β_1, β_2-inverse-strongly monotone mappings, we have

$$\begin{aligned} \|(I - \lambda_1 A_1)x - (I - \lambda_1 A_1)y\|^2 &= \|(x - y) - \lambda_1(A_1 x - A_1 y)\|^2 \\ &= \|x - y\|^2 - 2\lambda_1\langle x - y, A_1 x - A_1 y\rangle + \lambda_1^2 \|A_1 x - A_1 y\|^2 \\ &\leq \|x - y\|^2 + \lambda_1(\lambda_1 - 2\beta_1)\|A_1 x - A_1 y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

In similar way, we can obtain

$$\|(I - \lambda_2 A_2)x - (I - \lambda_2 A_2)y\|^2 \leq \|x - y\|^2.$$

And B_1, B_2 are η, ρ-inverse-strongly monotone mappings, we have

$$\begin{aligned}\left\|(I-r_nB_1)x-(I-r_nB_1)y\right\|^2 &= \left\|(x-y)-r_n(B_1x-B_1y)\right\|^2\\ &= \left\|x-y\right\|^2-2r_n\left\langle x-y,B_1x-B_1y\right\rangle+r_n^2\left\|B_1x-B_1y\right\|^2\\ &\le \left\|x-y\right\|^2+r_n(r_n-2\eta)\left\|B_1x-B_1y\right\|^2\\ &\le \left\|x-y\right\|^2.\end{aligned}$$

In similar way, we can obtain

$$\left\|(I-s_nB_2)x-(I-s_nB_2)y\right\|^2\le\left\|x-y\right\|^2.$$

It is clear that if $0 <\lambda_1 < 2\beta_1$, $0 <\lambda_2 < 2\beta_2$, $0 <r_n < 2\eta$, $0 <s_n \le 2\rho$ then $I - \lambda_1 A_1$, $I - \lambda_2 A_2$, $I - r_nB_1$, $I - s_nB_2$ are all nonexpansive. We will divide the proof into six steps.

Step 1. We will show $\{x_n\}$ is bounded. Put $y_n=J_{M_1,\lambda_1}(u_n-\lambda_1A_1u_n)$ for all $n \ge 0$ and $w_n=J_{M_2,\lambda_2}(v_n-\lambda_2A_2v_n)$ for all $n \ge 0$. It follows that

$$\begin{aligned}\left\|y_n-q\right\| &= \left\|J_{M_1,\lambda_1}(u_n-\lambda_1A_1u_n)-J_{M_1,\lambda_1}(q-\lambda_1A_1q)\right\|\\ &\le \left\|u_n-q\right\|.\end{aligned}\tag{3.3}$$

In similar way, we can obtain

$$\begin{aligned}\left\|w_n-q\right\| &= \left\|J_{M_2,\lambda_2}(v_n-\lambda_2A_2v_n)-J_{M_2,\lambda_2}(q-\lambda_2A_2q)\right\|\\ &\le \left\|v_n-q\right\|.\end{aligned}\tag{3.4}$$

Put $y'_n=P_Cy_n,\ n\ge 0$. It follows that

$$\begin{aligned}\left\|y'_n-q\right\| &= \left\|P_Cy_n-P_Cq\right\|\\ &\le \left\|y_n-q\right\|.\end{aligned}\tag{3.5}$$

By Lemma 2.7, we have $u_n=T_{r_n}^{(F_1,\varphi_1)}(x_n-r_nB_1x_n), v_n=T_{s_n}^{(F_2,\varphi_2)}(x_n-s_nB_2x_n)$ for all $n \ge 0$. Then, we have

$$\begin{aligned}\left\|u_n-q\right\|^2 &= \left\|T_{r_n}^{(F_1,\varphi_1)}(x_n-r_nB_1x_n)-T_{r_n}^{(F_1,\varphi_1)}(q-r_nB_1q)\right\|^2\\ &\le \left\|(x_n-r_nB_1x_n)-(q-r_nB_1q)\right\|^2\\ &\le \left\|x_n-q\right\|^2+r_n(r_n-2\eta)\left\|B_1x_n-B_1q\right\|^2\\ &\le \left\|x_n-q\right\|^2.\end{aligned}\tag{3.6}$$

In similar way, we can obtain

$$\begin{aligned}\left\|v_n-q\right\|^2 &= \left\|T_{s_n}^{(F_2,\varphi_2)}(x_n-s_nB_2x_n)-T_{s_n}^{(F_2,\varphi_2)}(q-s_nB_2q)\right\|^2\\ &\le \left\|(x_n-s_nB_2x_n)-(q-s_nB_2q)\right\|^2\\ &\le \left\|x_n-q\right\|^2+s_n(s_n-2\rho)\left\|B_2x_n-B_2q\right\|^2\\ &\le \left\|x_n-q\right\|^2.\end{aligned}\tag{3.7}$$

Put $z_n = P_C[\alpha_n\gamma f(x_n) + (I - \alpha_nA)SP_Cy_n]$ for all $n \ge 0$. From (3.1) and by Lemma 2.9

(ii), we deduce that

$$\begin{aligned}
\|x_{n+1}-q\| &= \|\xi_n(z_n-q)+(1-\xi_n)(P_Cw_n-q)\| \\
&\le \xi_n\|P_C[\alpha_n\gamma f(x_n)+(I-\alpha_nA)SP_Cy_n]-P_Cq\|+(1-\xi_n)\|P_Cw_n-P_Cq\| \\
&\le \xi_n\|\alpha_n\gamma f(x_n)+(I-\alpha_nA)SP_Cy_n-q\|+(1-\xi_n)\|w_n-q\| \\
&= \xi_n\|\alpha_n(\gamma f(x_n)-Aq)+(I-\alpha_nA)(SP_Cy_n-q)\|+(1-\xi_n)\|w_n-q\| \\
&\le \xi_n\alpha_n\|\gamma f(x_n)-Aq\| \\
&\quad+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|P_Cy_n-q\|+(1-\xi_n)\|w_n-q\| \\
&\le \xi_n\alpha_n\|\gamma f(x_n)-\gamma f(q)\|+\xi_n\alpha_n\|\gamma f(q)-Aq\| \\
&\quad+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|P_Cy_n-q\|+(1-\xi_n)\|x_n-q\| \\
&\le \xi_n\alpha_n\gamma\alpha\|x_n-q\|+\xi_n\alpha_n\|\gamma f(q)-Aq\| \\
&\quad+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_n-q\|+(1-\xi_n)\|x_n-q\| \\
&= \left(1-\left(\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)-\gamma\alpha\right)\xi_n\alpha_n\right)\|x_n-q\|+\xi_n\alpha_n\|\gamma f(q)-Aq\| \\
&\le \left(1-\left(\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)-\gamma\alpha\right)\xi_n\alpha_n\right)\|x_n-q\| \\
&\quad+\left(\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)-\gamma\alpha\right)\xi_n\alpha_n\frac{\|\gamma f(q)-Aq\|}{\left(\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)-\gamma\alpha\right)} \\
&\le \max\left\{\|x_n-q\|,\frac{\|\gamma f(q)-Aq\|}{\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)-\gamma\alpha}\right\}.
\end{aligned} \tag{3.8}$$

It follows from induction that

$$\|x_n-q\|\le\max\left\{\|x_0-q\|,\frac{\|\gamma f(q)-Aq\|}{\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)-\gamma\alpha}\right\},\quad n\ge 0.$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\}$,$\{z_n\}$,$\{P_Cw_n\}$,$\{SP_Cy_n\}$,$\{f(x_n)\}$ and $\{ASP_Cy_n\}$.
Step 2. We claim that $\lim_{n\to\infty}||x_{n+2}-x_{n+1}||=0$. From (3.1), we have

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\xi_{n+1}z_{n+1}+(1-\xi_{n+1})P_Cw_{n+1}-\xi_nz_n-(1-\xi_n)P_Cw_n\| \\
&= \|\xi_{n+1}(z_{n+1}-z_n)+(\xi_{n+1}-\xi_n)z_n \\
&\quad+(1-\xi_{n+1})(P_Cw_{n+1}-P_Cw_n)+(\xi_n-\xi_{n+1})P_Cw_n\| \\
&\le \xi_{n+1}\|z_{n+1}-z_n\|+(1-\xi_{n+1})\|w_{n+1}-w_n\| \\
&\quad+|\xi_{n+1}-\xi_n|(\|z_n\|+\|P_Cw_n\|).
\end{aligned} \tag{3.9}$$

Since $I - \lambda_2 A_2$ be nonexpansive, we have

$$\begin{aligned}\|w_{n+1} - w_n\| &= \left\|J_{M_2,\lambda_2}(v_{n+1} - \lambda_2 A_2 v_{n+1}) - J_{M_2,\lambda_2}(v_n - \lambda_2 A_2 v_n)\right\| \\ &\le \left\|(v_{n+1} - \lambda_2 A_2 v_{n+1}) - (v_n - \lambda_2 A_2 v_n)\right\| \\ &\le \|v_{n+1} - v_n\|.\end{aligned} \tag{3.10}$$

On the other hand, from $v_{n-1} = T_{s_{n-1}}^{(F_2,\varphi_2)}(x_{n-1} - s_{n-1}B_2 x_{n-1})$ and $v_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B_2 x_n)$, it follows that

$$\begin{aligned}&F_2(v_{n-1}, y) + \langle B_2 x_{n-1}, y - v_{n-1}\rangle + \varphi_2(y) - \varphi_2(v_{n-1}) \\ &\quad + \frac{1}{s_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C\end{aligned} \tag{3.11}$$

and

$$F_2(v_n, y) + \langle B_2 x_n, y - v_n\rangle + \varphi_2(y) - \varphi_2(v_n) + \frac{1}{s_n}\langle y - v_n, v_n - x_n\rangle \ge 0, \quad \forall y \in C \tag{3.12}$$

Substituting $y = v_n$ in (3.11) and $y = v_{n-1}$ in (3.12), we get

$$F_2(v_{n-1}, v_n) + \langle B_2 x_{n-1}, v_n - v_{n-1}\rangle + \varphi_2(v_n) - \varphi_2(v_{n-1}) + \frac{1}{s_{n-1}}\langle v_n - v_{n-1}, v_{n-1} - x_{n-1}\rangle \ge 0$$

and

$$F_2(v_n, v_{n-1}) + \langle B_2 x_n, v_{n-1} - v_n\rangle + \varphi_2(v_{n-1}) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_{n-1} - v_n, v_n - x_n\rangle \ge 0.$$

From (A2), we obtain

$$\left\langle v_n - v_{n-1}, B_2 x_{n-1} - B_2 x_n + \frac{v_{n-1} - x_{n-1}}{s_{n-1}} - \frac{v_n - x_n}{s_n}\right\rangle \ge 0,$$

and then

$$\left\langle v_n - v_{n-1}, s_{n-1}(B_2 x_{n-1} - B_2 x_n) + v_{n-1} - x_{n-1} - \frac{s_{n-1}}{s_n}(v_n - x_n)\right\rangle \ge 0,$$

so

$$\left\langle v_n - v_{n-1}, s_{n-1}B_2 x_{n-1} - s_{n-1}B_2 x_n + v_{n-1} - v_n + v_n - x_{n-1} - \frac{s_{n-1}}{s_n}(v_n - x_n)\right\rangle \ge 0.$$

It follows that

$$\begin{gathered}\Big\langle v_n - v_{n-1}, (I - s_{n-1}B_2)x_n - (I - s_{n-1}B_2)x_{n-1} \\ + v_{n-1} - v_n + v_n - x_n - \frac{s_{n-1}}{s_n}(v_n - x_n)\Big\rangle \ge 0, \\ \langle v_n - v_{n-1}, v_{n-1} - v_n\rangle + \left\langle v_n - v_{n-1}, x_n - x_{n-1} + \left(1 - \frac{s_{n-1}}{s_n}\right)(v_n - x_n)\right\rangle \ge 0.\end{gathered}$$

Without loss of generality, let us assume that there exists a real number e such that $s_{n-1} > e > 0$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|v_n - v_{n-1}\|^2 &\le \left\langle v_n - v_{n-1}, x_n - x_{n-1} + \left(1 - \frac{s_{n-1}}{s_n}\right)(v_n - x_n)\right\rangle \\ &\le \|v_n - v_{n-1}\| \left\{\|x_n - x_{n-1}\| + \left|1 - \frac{s_{n-1}}{s_n}\right| \|v_n - x_n\|\right\}\end{aligned}$$

and hence

$$\begin{aligned}\|v_n - v_{n-1}\| &\le \|x_n - x_{n-1}\| + \frac{1}{s_n}|s_n - s_{n-1}|\,\|v_n - x_n\| \\ &\le \|x_n - x_{n-1}\| + \frac{M_1}{e}|s_n - s_{n-1}|,\end{aligned} \tag{3.13}$$

where $M_1 = \sup\{||v_n - x_n|| : n \in \mathbb{N}\}$. Substituting (3.13) into (3.9) and (3.10) that

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| \le \xi_{n+1}\|z_{n+1} - z_n\| &+ (1 - \xi_{n+1})\left\{\|x_{n+1} - x_n\| + \frac{M_1}{e}|s_n - s_{n-1}|\right\} \\ &+ |\xi_{n+1} - \xi_n|(\|z_n\| + \|P_C w_n\|).\end{aligned} \tag{3.14}$$

By Lemma 2.9 (ii), it follow that

$$\begin{aligned}&\|z_{n+1} - z_n\| \\ &= \big\|P_C[\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)SP_C y_{n+1}] \\ &\quad - P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)SP_C y_n]\big\| \\ &\le \big\|\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)SP_C y_{n+1} - \alpha_n\gamma f(x_n) - (I - \alpha_n A)SP_C y_n\big\| \\ &= \big\|\alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)\gamma f(x_n) \\ &\quad + (I - \alpha_{n+1}A)(SP_C y_{n+1} - SP_C y_n) + (\alpha_n - \alpha_{n+1})ASP_C y_n\big\| \\ &\le \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\,\|\gamma f(x_n)\| \\ &\quad + \left(1 - \alpha_{n+1}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_{n+1} - y_n\| + |\alpha_{n+1} - \alpha_n|\,\|ASP_C y_n\| \\ &= \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\left(\|\gamma f(x_n)\| + \|ASP_C y_n\|\right) \\ &\quad + \left(1 - \alpha_{n+1}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_{n+1} - y_n\|.\end{aligned} \tag{3.15}$$

Since $I - \lambda_1 A_1$ be nonexpansive, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|J_{M_1,\lambda_1}(u_{n+1} - \lambda_1 A_1 u_{n+1}) - J_{M_1,\lambda_1}(u_n - \lambda_1 A_1 u_n)\| \\ &\le \|(u_{n+1} - \lambda_1 A_1 u_{n+1}) - (u_n - \lambda_1 A_1 u_n)\| \\ &\le \|(I - \lambda_1 A_1)u_{n+1} - (I - \lambda_1 A_1)u_n\| \\ &\le \|u_{n+1} - u_n\|.\end{aligned} \tag{3.16}$$

On the other hand, from $u_{n-1} = T_{r_{n-1}}^{(F_1,\varphi_1)}(x_{n-1} - r_{n-1}B_1 x_{n-1})$ and $u_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n B_1 x_n)$, it follows that

$$\begin{aligned}&F_1(u_{n-1}, y) + \langle B_1 x_{n-1}, y - u_{n-1}\rangle + \varphi_1(y) - \varphi_1(u_{n-1}) \\ &\quad + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C\end{aligned} \tag{3.17}$$

and

$$F_1(u_n, y) + \langle B_1 x_n, y - u_n \rangle + \varphi_1(y) - \varphi_1(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \quad (3.18)$$

Substituting $y = u_n$ in (3.17) and $y = u_{n-1}$ in (3.18), we get

$$F_1(u_{n-1}, u_n) + \langle B_1 x_{n-1}, u_n - u_{n-1} \rangle + \varphi_1(u_n) - \varphi_1(u_{n-1}) + \frac{1}{r_{n-1}} \langle u_n - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0$$

and

$$F_1(u_n, u_{n-1}) + \langle B_1 x_n, u_{n-1} - u_n \rangle + \varphi_1(u_{n-1}) - \varphi_1(u_n) + \frac{1}{r_n} \langle u_{n-1} - u_n, u_n - x_n \rangle \geq 0.$$

From (A2), we obtain

$$\left\langle u_n - u_{n-1}, B_1 x_{n-1} - B_1 x_n + \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n} \right\rangle \geq 0,$$

and then

$$\left\langle u_n - u_{n-1}, r_{n-1}(B_1 x_{n-1} - B_1 x_n) + u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n) \right\rangle \geq 0,$$

so

$$\left\langle u_n - u_{n-1}, r_{n-1} B_1 x_{n-1} - r_{n-1} B_1 x_n + u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n) \right\rangle \geq 0.$$

It follows that

$$\begin{gathered} \Big\langle u_n - u_{n-1}, (I - r_{n-1} B_1) x_n - (I - r_{n-1} B_1) x_{n-1} \\ + u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}(u_n - x_n) \Big\rangle \geq 0, \\ \langle u_n - u_{n-1}, u_{n-1} - u_n \rangle + \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n) \right\rangle \geq 0. \end{gathered}$$

Without loss of generality, let us assume that there exists a real number c such that $r_{n-1} > c > 0$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 &\leq \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n) \right\rangle \\ &\leq \|u_n - u_{n-1}\| \left\{ \|x_n - x_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right| \|u_n - x_n\| \right\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{r_n} |r_n - r_{n-1}| \, \|u_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{M_2}{c} |r_n - r_{n-1}|, \end{aligned} \quad (3.19)$$

where $M_2 = \sup\{||u_n - x_n|| : n \in \mathbb{N}\}$. Substituting (3.19) into (3.16), we have

$$\left\|y_n - y_{n-1}\right\| \le \|x_n - x_{n-1}\| + \frac{M_2}{c}\,|r_n - r_{n-1}|\,. \tag{3.20}$$

Substituting (3.20) into (3.15), we obtain that

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \alpha_{n+1}\gamma\alpha\,\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\left(\left\|\gamma f(x_n)\right\| + \left\|ASP_C y_n\right\|\right)\\ &\quad + \left(1 - \alpha_{n+1}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\{\|x_n - x_{n-1}\| + \frac{M_2}{c}\,|r_n - r_{n-1}|\right\}\end{aligned} \tag{3.21}$$

And substituting (3.10), (3.13), (3.21) into (3.9), we get

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &\le \xi_{n+1}\left\{\alpha_{n+1}\gamma\alpha\,\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\left(\left\|\gamma f(x_n)\right\| + \left\|ASP_C y_n\right\|\right)\right.\\ &\quad \left. + \left(1 - \alpha_{n+1}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|x_n - x_{n-1}\| + \frac{M_2}{c}\,|r_n - r_{n-1}|\right\}\\ &\quad + (1 - \xi_{n+1})\left\{\|x_n - x_{n-1}\| + \frac{M_1}{e}\,|s_n - s_{n-1}|\right\}\\ &\quad + |\xi_{n+1} - \xi_n|\left(\|z_n\| + \|P_C w_n\|\right)\\ &\le \left(1 - \left(\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) - \gamma\alpha\right)\xi_{n+1}\alpha_{n+1}\right)\|x_{n+1} - x_n\|\\ &\quad + \left(|\alpha_{n+1} - \alpha_n| + |\xi_{n+1} - \xi_n|\right)M_3\\ &\quad + \frac{M_2}{c}\,|r_n - r_{n-1}| + \frac{M_1}{e}\,|s_n - s_{n-1}|\,,\end{aligned} \tag{3.22}$$

where $M_3 > 0$ is a constant satisfying

$$\sup_n\left\{\left\|\gamma f(x_n)\right\| + \left\|ASP_C y_n\right\|, \|z_n\| + \|P_C w_n\|\right\} \le M_3.$$

This together with (C1)-(C4) and Lemma 2.5, imply that

$$\lim_{n\to\infty}\|x_{n+2} - x_{n+1}\| = 0. \tag{3.23}$$

From (3.20) and (C3), we also have $||y_{n+1} - y_n|| \to 0$ as $n \to \infty$.

Step 3. We show the followings:

(i) $\lim_{n\to\infty} ||A_1u_n - A_1q|| = 0$;

(ii) $\lim_{n\to\infty} ||A_2v_n - A_2q|| = 0$;

(iii) $\lim_{n\to\infty} ||B_1x_n - B_1q|| = 0$;

(iv) $\lim_{n\to\infty} ||B_2x_n - B_2q|| = 0$.

For $q \in \Theta$ and $q = J_{M_1,\lambda_1}(q - \lambda_1A_1q)$, then we get

$$\begin{aligned}\left\|y_n - q\right\|^2 &= \left\|J_{M_1,\lambda_1}(u_n - \lambda_1A_1u_n) - J_{M_1,\lambda_1}(q - \lambda_1A_1q)\right\|^2\\ &\le \left\|(u_n - \lambda_1A_1u_n) - (q - \lambda_1A_1q)\right\|^2\\ &\le \left\|u_n - q\right\|^2 + \lambda_1(\lambda_1 - 2\beta_1)\left\|A_1u_n - A_1q\right\|^2\\ &\le \left\|x_n - q\right\|^2 + \lambda_1(\lambda_1 - 2\beta_1)\left\|A_1u_n - A_1q\right\|^2.\end{aligned}$$

Using (3.5), it follows that

$$\begin{aligned}
&\|z_n - q\|^2 \\
&\quad = \|P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)S{y'}_n] - P_C(q)\|^2 \\
&\quad \le \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(S{y'}_n - q)\|^2 \\
&\quad \le \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)({y'}_n - q)\|^2 \\
&\quad \le \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(y_n - q)\|^2 \\
&\quad \le \alpha_n\|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_n - q\|^2 \\
&\qquad + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\|\,\|y_n - q\| \\
&\quad \le \alpha_n\|\gamma f(x_n) - Aq\|^2 + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\|\,\|y_n - q\| \\
&\qquad + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\{\|x_n - q\|^2 + \lambda_1(\lambda_1 - 2\beta_1)\|A_1u_n - A_1q\|^2\right\} \\
&\quad \le \alpha_n\|\gamma f(x_n) - Aq\|^2 + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\|\,\|y_n - q\| \\
&\qquad + \|x_n - q\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1(\lambda_1 - 2\beta_1)\|A_1u_n - A_1q\|^2.
\end{aligned} \tag{3.24}$$

By the convexity of the norm $||\cdot||$, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\xi_n z_n + (1 - \xi_n)P_C w_n - q\|^2 \\
&= \|\xi_n(z_n - q) + (1 - \xi_n)(P_C w_n - q)\|^2 \\
&\le \xi_n\|z_n - q\|^2 + (1 - \xi_n)\|w_n - q\|^2.
\end{aligned} \tag{3.25}$$

Substituting (3.4), (3.7), (3.24) into (3.25), we obtain

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \le \xi_n\left\{\alpha_n\|\gamma f(x_n) - Aq\|^2 + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\right. \\
&\qquad \times \|\gamma f(x_n) - Aq\|\,\|y_n - q\| + \|x_n - q\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \\
&\qquad \left.\times\lambda_1(\lambda_1 - 2\beta_1)\|A_1u_n - A_1q\|^2\right\} + (1 - \xi_n)\|x_n - q\|^2 \\
&\quad = \xi_n\alpha_n\|\gamma f(x_n) - Aq\|^2 + 2\xi_n\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \\
&\qquad \times \|\gamma f(x_n) - Aq\|\,\|y_n - q\| + \xi_n\|x_n - q\|^2 + \xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \\
&\qquad \times \lambda_1(\lambda_1 - 2\beta_1)\|A_1u_n - A_1q\|^2 + (1 - \xi_n)\|x_n - q\|^2.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\xi_n\left(1 - \alpha_n\left(1 - \frac{1-\delta}{\mu}\right)\right)\lambda_1(2\beta_1 - \lambda_1)\|A_1u_n - A_1q\|^2 \\
&\quad \le \xi_n\alpha_n\|\gamma f(x_n) - Aq\|^2 + \varepsilon_n + \|x_n - x_{n+1}\|\left(\|x_n - q\| + \|x_{n+1} - q\|\right),
\end{aligned}$$

where $\varepsilon_n = 2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\,\|y_n-q\|$. Since conditions (C1), (C2) and $\lim_{n\to\infty}||x_{n+1}\text{-}x_n|| = 0$, then we obtain that $||A_1u_n - A_1q|| \to 0$ as $n \to \infty$. For $q \in \Theta$ and $q = J_{M_2,\lambda_2}(q-\lambda_2A_2q)$, then we get

$$
\begin{aligned}
\|w_n-q\|^2 &= \|J_{M_2,\lambda_2}(v_n-\lambda_2A_2v_n)-J_{M_2,\lambda_2}(q-\lambda_2A_2q)\|^2\\
&\le \|(v_n-\lambda_2A_2v_n)-(q-\lambda_2A_2q)\|^2\\
&\le \|v_n-q\|^2+\lambda_2(\lambda_2-2\beta_2)\|A_2v_n-A_2q\|^2\\
&\le \|x_n-q\|^2+\lambda_2(\lambda_2-2\beta_2)\|A_2v_n-A_2q\|^2.
\end{aligned}
\tag{3.26}
$$

Substituting (3.24), (3.26) into (3.25), we obtain

$$
\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\le \xi_n\Bigg\{\alpha_n\|\gamma f(x_n)-Aq\|^2+2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\,\|y_n-q\|\\
&\quad+\|x_n-q\|^2+\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1(\lambda_1-2\beta_1)\|A_1u_n-A_1q\|^2\Bigg\}\\
&\quad+(1-\xi_n)\left\{\|x_n-q\|^2+\lambda_2(\lambda_2-2\beta_2)\|A_2v_n-A_2q\|^2\right\}\\
&= \xi_n\alpha_n\|\gamma f(x_n)-Aq\|^2+2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\,\|y_n-q\|\\
&\quad+\xi_n\|x_n-q\|^2+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1(\lambda_1-2\beta_1)\|A_1u_n-A_1q\|^2\\
&\quad+(1-\xi_1)\|x_n-q\|^2+(1-\xi_n)\lambda_2(\lambda_2-2\beta_2)\|A_2v_n-A_2q\|^2.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&(1-\xi_n)\lambda_2(2\beta_2-\lambda_2)\|A_2v_n-A_2q\|^2\\
&\le \xi_n\alpha_n\|\gamma f(x_n)-Aq\|^2+\varepsilon_n+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1(\lambda_1-2\beta_1)\\
&\quad\times\|A_1u_n-A_1q\|^2+\|x_n-x_{n+1}\|\left(\|x_n-q\|+\|x_{n+1}-q\|\right),
\end{aligned}
$$

where $\varepsilon_n = 2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\,\|y_n-q\|$. Since conditions (C1), (C2), $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$ and $\lim_{n\to\infty}||A_1u_n - A_1q|| \to 0$ then we obtain that $||A_2v_n - A_2q|| \to 0$ as $n \to \infty$. We consider this inequality in (3.24) that

$$
\begin{aligned}
\|z_n-q\|^2 &\le \alpha_n\|\gamma f(x_n)-Aq\|^2+\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_n-q\|^2\\
&\quad+2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\,\|y_n-q\|.
\end{aligned}
\tag{3.27}
$$

Substituting (3.3) and (3.6) into (3.27), we have

$$\begin{aligned}
\|z_n - q\|^2 &\le \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \\
&\quad \times \left\{\|x_n - q\|^2 + r_n(r_n - 2\eta)\|B_1x_n - B_1q\|^2\right\} \\
&\quad + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\|\,\|y_n - q\| \\
&= \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|x_n - q\|^2 \\
&\quad + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n(r_n - 2\eta)\|B_1x_n - B_1q\|^2 \\
&\quad + 2\alpha_n\left(1 + \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\|\,\|y_n - q\| \\
&\le \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 \\
&\quad + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n(r_n - 2\eta)\|B_1x_n - B_1q\|^2 \\
&\quad + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\|\,\|y_n - q\|.
\end{aligned} \tag{3.28}$$

Substituting (3.4), (3.7) and (3.28) into (3.25), we obtain

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\le \xi_n\left\{\alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\right. \\
&\quad \times r_n(r_n - 2\eta)\|B_1x_n - B_1q\|^2 + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \\
&\quad \times \|\gamma f(x_n) - Aq\|\,\|y_n - q\|\} + (1 - \xi_n)\|x_n - q\|^2 \\
&= \xi_n\alpha_n\|\gamma f(x_n) - Aq\|^2 + \xi_n\|x_n - q\|^2 + \xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \\
&\quad \times r_n(r_n - 2\eta)\|B_1x_n - B_1q\|^2 + 2\xi_n\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \\
&\quad \times \|\gamma f(x_n) - Aq\|\,\|y_n - q\| + (1 - \xi_n)\|x_n - q\|^2.
\end{aligned} \tag{3.29}$$

So, we also have

$$\begin{aligned}
&\xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n(2\eta - r_n)\|B_1x_n - B_1q\|^2 \\
&\quad \le \xi_n\alpha_n\|\gamma f(x_n) - Aq\|^2 + \varepsilon_n + \|x_n - x_{n+1}\|\left(\|x_n - q\| + \|x_{n+1} - q\|\right),
\end{aligned}$$

where $\varepsilon_n = 2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\left\|y_n-q\right\|$. Since conditions (C1), (C2), (C3), $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$, then we obtain that $||B_1x_n - B_1q|| \to 0$ as $n \to \infty$. Substituting (3.4), (3.7) and (3.28) into (3.25), we obtain

$$\begin{aligned}
&\left\|x_{n+1}-q\right\|^2\\
&\le \xi_n\left\{\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2+\left\|x_n-q\right\|^2+\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\right.\\
&\quad\times r_n(r_n-2\eta)\left\|B_1x_n-B_1q\right\|^2+2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\\
&\quad\left.\times\left\|\gamma f(x_n)-Aq\right\|\left\|y_n-q\right\|\right\}\\
&\quad+(1-\xi_n)\{\left\|x_n-q\right\|^2+s_n(s_n-2\rho)\left\|B_2x_n-B_2q\right\|^2\}\\
&=\xi_n\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2+\xi_n\left\|x_n-q\right\|^2+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\\
&\quad\times r_n(r_n-2\eta)\left\|B_1x_n-B_1q\right\|^2+2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\\
&\quad\times\left\|\gamma f(x_n)-Aq\right\|\left\|y_n-q\right\|+(1-\xi_n)\left\|x_n-q\right\|^2\\
&\quad+(1-\xi_n)s_n(s_n-2\rho)\left\|B_2x_n-B_2q\right\|^2\\
&=\xi_n\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2+\left\|x_n-q\right\|^2+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\\
&\quad\times r_n(r_n-2\eta)\left\|B_1x_n-B_1q\right\|^2+2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\\
&\quad\times\left\|\gamma f(x_n)-Aq\right\|\left\|y_n-q\right\|\\
&\quad+(1-\xi_n)s_n(s_n-2\rho)\left\|B_2x_n-B_2q\right\|^2.
\end{aligned}$$

So, we also have

$$\begin{aligned}
&(1-\xi_n)s_n(2\rho-s_n)\left\|B_2x_n-B_2q\right\|^2\\
&\quad\le\xi_n\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2+\varepsilon_n+\|x_n-x_{n+1}\|\left(\left\|x_n-q\right\|+\left\|x_{n+1}-q\right\|\right)\\
&\qquad+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n(r_n-2\eta)\left\|B_1x_n-B_1q\right\|^2,
\end{aligned}$$

where $\varepsilon_n = 2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\left\|y_n-q\right\|$. Since conditions (C1), (C2), (C4), $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$ and $\lim_{n\to\infty}||B_1x_n - B_1q|| = 0$, then we obtain that $||B_2x_n - B_2q|| \to 0$ as $n \to \infty$.

Step 4. We show the followings:

(i) $\lim_{n\to\infty}||x_n - u_n|| = 0$;

(ii) $\lim_{n\to\infty}||u_n - y_n|| = 0$;

(iii) $\lim_{n\to\infty}||x_n - v_n|| = 0$;

(iv) $\lim_{n\to\infty}||v_n - w_n|| = 0$;

(v) $\lim_{n\to\infty} \left\| u_n - y'_n \right\| = 0$;

(vi) $\lim_{n\to\infty} \left\| y'_n - Sy'_n \right\| = 0$.

Since $T_{r_n}^{(F_1,\varphi_1)}$ is firmly nonexpansive, we observe that

$$\begin{aligned}
&\left\| u_n - q \right\|^2 \\
&\quad = \left\| T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nB_1x_n) - T_{r_n}^{(F_1,\varphi_1)}(q - r_nB_1q) \right\|^2 \\
&\quad \le \left\langle (x_n - r_nB_1x_n) - (q - r_nB_1q), u_n - q \right\rangle \\
&\quad = \frac{1}{2}\Big(\left\| (x_n - r_nB_1x_n) - (q - r_nB_1q) \right\|^2 + \left\| u_n - q \right\|^2 - \big\| (x_n - r_nB_1x_n) \\
&\qquad - (q - r_nB_1q) - (u_n - q) \big\|^2 \Big) \\
&\quad \le \frac{1}{2}\Big(\left\| x_n - q \right\|^2 + \left\| u_n - q \right\|^2 - \left\| (x_n - u_n) - r_n(B_1x_n - B_1q) \right\|^2 \Big) \\
&\quad = \frac{1}{2}\Big(\left\| x_n - q \right\|^2 + \left\| u_n - q \right\|^2 - \| x_n - u_n \|^2 \\
&\qquad + 2r_n \left\langle B_1x_n - B_1q, x_n - u_n \right\rangle - r_n^2 \left\| B_1x_n - B_1q \right\|^2 \Big).
\end{aligned}$$

Hence, we have

$$\left\| u_n - q \right\|^2 \le \left\| x_n - q \right\|^2 - \| x_n - u_n \|^2 + 2r_n \left\| B_1x_n - B_1q \right\| \| x_n - u_n \| . \tag{3.30}$$

Since J_{M_1,λ_1} is 1-inverse-strongly monotone, we compute

$$\begin{aligned}
\left\| y_n - q \right\|^2 &= \left\| J_{M_1,\lambda_1}(u_n - \lambda_1A_1u_n) - J_{M_1,\lambda_1}(q - \lambda_1A_1q) \right\|^2 \\
&\le \left\langle (u_n - \lambda_1A_1u_n) - (q - \lambda_1A_1q), y_n - q \right\rangle \\
&= \frac{1}{2}\Big(\left\| (u_n - \lambda_1A_1u_n) - (q - \lambda_1A_1q) \right\|^2 + \left\| y_n - q \right\|^2 \\
&\quad - \left\| (u_n - \lambda_1A_1u_n) - (q - \lambda_1A_1q) - (y_n - q) \right\|^2 \\
&= \frac{1}{2}\Big(\left\| u_n - q \right\|^2 + \left\| y_n - q \right\|^2 - \left\| (u_n - y_n) - \lambda_1(A_1u_n - A_1q) \right\|^2 \Big) \\
&\le \frac{1}{2}\Big(\left\| u_n - q \right\|^2 + \left\| y_n - q \right\|^2 - \left\| u_n - y_n \right\|^2 \\
&\quad + 2\lambda_1 \left\langle u_n - y_n, A_1u_n - A_1q \right\rangle - \lambda_1^2 \left\| A_1u_n - A_1q \right\|^2 \Big),
\end{aligned}$$

which implies that

$$\left\| y_n - q \right\|^2 \le \left\| u_n - q \right\|^2 - \left\| u_n - y_n \right\|^2 + 2\lambda_1 \left\| u_n - y_n \right\| \left\| A_1u_n - A_1q \right\| . \tag{3.31}$$

Substitute (3.30) into (3.31), we have

$$\begin{aligned}
\left\| y_n - q \right\|^2 &\le \{ \left\| x_n - q \right\|^2 - \| x_n - u_n \|^2 + 2r_n \left\| B_1x_n - B_1q \right\| \| x_n - u_n \| \} \\
&\quad - \left\| u_n - y_n \right\|^2 + 2\lambda_1 \left\| u_n - y_n \right\| \left\| A_1u_n - A_1q \right\| .
\end{aligned} \tag{3.32}$$

Substitute (3.32) into (3.27), we have

$$\begin{aligned}
\|z_n - q\|^2 &\le \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\{\|x_n - q\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r_n \|B_1x_n - B_1q\| \|x_n - u_n\| - \|u_n - y_n\|^2 + 2\lambda_1 \|u_n - y_n\| \|A_1u_n - A_1q\|\} \\
&\quad + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\| \|y_n - q\| \\
&\le \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n \|B_1x_n - B_1q\| \|x_n - u_n\| - \|u_n - y_n\|^2 \\
&\quad + 2\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1 \|u_n - y_n\| \|A_1u_n - A_1q\| \\
&\quad + 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\| \|y_n - q\|.
\end{aligned} \tag{3.33}$$

Since $T_{s_n}^{(F_2,\varphi_2)}$ is firmly nonexpansive, we observe that

$$\begin{aligned}
&\|v_n - q\|^2 \\
&= \left\|T_{s_n}^{(F_2,\varphi_2)}(x_n - s_nB_2x_n) - T_{s_n}^{(F_2,\varphi_2)}(q - s_nB_2q)\right\|^2 \\
&\le \langle (x_n - s_nB_2x_n) - (q - s_nB_2q), v_n - q\rangle \\
&= \frac{1}{2}(\|(x_n - s_nB_2x_n) - (q - s_nB_2q)\|^2 + \|v_n - q\|^2 \\
&\quad - \|(x_n - s_nB_2x_n) - (q - s_nB_2q) - (v_n - q)\|^2) \\
&\le \frac{1}{2}(\|x_n - q\|^2 + \|v_n - q\|^2 - \|(x_n - v_n) - s_n(B_2x_n - B_2q)\|^2) \\
&= \frac{1}{2}(\|x_n - q\|^2 + \|v_n - q\|^2 - \|x_n - v_n\|^2 \\
&\quad + 2s_n\langle B_2x_n - B_2q, x_n - v_n\rangle - s_n^2\|B_2x_n - B_2q\|^2).
\end{aligned}$$

Hence, we have

$$\|v_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - v_n\|^2 + 2s_n\|B_2x_n - B_2q\| \|x_n - v_n\|. \tag{3.34}$$

Since J_{M_2,λ_2} is 1-inverse-strongly monotone, we compute

$$\begin{aligned}
\|w_n - q\|^2 &= \|J_{M_2,\lambda_2}(v_n - \lambda_2A_2v_n) - J_{M_2,\lambda_2}(q - \lambda_2A_2q)\|^2 \\
&\le \langle (v_n - \lambda_2A_2v_n) - (q - \lambda_2A_2q), w_n - q\rangle \\
&= \frac{1}{2}(\|(v_n - \lambda_2A_2v_n) - (q - \lambda_2A_2q)\|^2 + \|w_n - q\|^2 \\
&\quad - \|(v_n - \lambda_2A_2v_n) - (q - \lambda_2A_2q) - (w_n - q)\|^2) \\
&= \frac{1}{2}(\|v_n - q\|^2 + \|w_n - q\|^2 - \|(v_n - w_n) - \lambda_2(A_2v_n - A_2q)\|^2) \\
&\le \frac{1}{2}(\|v_n - q\|^2 + \|w_n - q\|^2 - \|v_n - w_n\|^2 \\
&\quad + 2\lambda_2\langle v_n - w_n, A_2v_n - A_2q\rangle - \lambda_2^2\|A_2v_n - A_2q\|^2),
\end{aligned}$$

which implies that

$$\|w_n - q\|^2 \le \|v_n - q\|^2 - \|v_n - w_n\|^2 + 2\lambda_2\|v_n - w_n\| \|A_2v_n - A_2q\|. \tag{3.35}$$

Substitute (3.34) into (3.35), we have

$$\begin{aligned}\left\|w_n - q\right\|^2 \le \{&\left\|x_n - q\right\|^2 - \|x_n - v_n\|^2 + 2s_n \left\|B_2x_n - B_2q\right\| \|x_n - v_n\|\} \\ &- \|v_n - w_n\|^2 + 2\lambda_2 \|v_n - w_n\| \left\|A_2v_n - A_2q\right\|.\end{aligned} \tag{3.36}$$

Substitute (3.33) and (3.36) into (3.25), we obtain

$$\begin{aligned}\left\|x_{n+1} - q\right\|^2 &\le \xi_n \left\|z_n - q\right\|^2 + (1-\xi_n)\left\|w_n - q\right\|^2 \\ &\le \xi_n \Bigg\{\alpha_n \left\|\gamma f(x_n) - Aq\right\|^2 + \left\|x_n - q\right\|^2 - \|x_n - u_n\|^2 - \left\|u_n - y_n\right\|^2 \\ &\quad + 2\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n \left\|B_1x_n - B_1q\right\| \|x_n - u_n\| \\ &\quad + 2\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda_1 \left\|u_n - y_n\right\| \left\|A_1u_n - A_1q\right\| \\ &\quad +2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \left\|\gamma f(x_n) - Aq\right\| \left\|y_n - q\right\|\Bigg\} \\ &\quad + (1-\xi_n)\{\left\|x_n - q\right\|^2 - \|x_n - v_n\|^2 \\ &\quad + 2s_n \left\|B_2x_n - B_2q\right\| \|x_n - v_n\| - \|v_n - w_n\|^2 \\ &\quad + 2\lambda_2 \|v_n - w_n\| \left\|A_2v_n - A_2q\right\|\} \\ &\le \xi_n\alpha_n \left\|\gamma f(x_n) - Aq\right\|^2 + \xi_n \left\|x_n - q\right\|^2 - \|x_n - u_n\|^2 - \left\|u_n - y_n\right\|^2 \\ &\quad + 2\xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n \left\|B_1x_n - B_1q\right\| \|x_n - u_n\| \\ &\quad + 2\xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda_1 \left\|u_n - y_n\right\| \left\|A_1u_n - A_1q\right\| \\ &\quad + 2\xi_n\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \left\|\gamma f(x_n) - Aq\right\| \left\|y_n - q\right\| \\ &\quad + (1-\xi_n)\left\|x_n - q\right\|^2 - \|x_n - v_n\|^2 + 2(1-\xi_n)s_n \left\|B_2x_n - B_2q\right\| \|x_n - v_n\| \\ &\quad - \|v_n - w_n\|^2 + 2(1-\xi_n)\lambda_2 \|v_n - w_n\| \left\|A_2v_n - A_2q\right\|.\end{aligned}$$

Then, we derive

$$\begin{aligned}&\|x_n - u_n\|^2 + \left\|u_n - y_n\right\|^2 + \|x_n - v_n\|^2 + \|v_n - w_n\|^2 \\ &\quad \le \xi_n\alpha_n \left\|\gamma f(x_n) - Aq\right\|^2 + \left\|x_n - q\right\|^2 - \left\|x_{n+1} - q\right\|^2 \\ &\qquad + 2\xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n \left\|B_1x_n - B_1q\right\| \|x_n - u_n\| \\ &\qquad + 2\xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda_1 \left\|u_n - y_n\right\| \left\|A_1u_n - A_1q\right\| \\ &\qquad + 2\xi_n\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \left\|\gamma f(x_n) - Aq\right\| \left\|y_n - q\right\| \\ &\qquad + 2(1-\xi_n)s_n \left\|B_2x_n - B_2q\right\| \|x_n - v_n\| + 2(1-\xi_n)\lambda_2 \|v_n - w_n\| \left\|A_2v_n - A_2q\right\| \\ &\quad = \xi_n\alpha_n \left\|\gamma f(x_n) - Aq\right\|^2 + \|x_{n+1} - x_n\| \left(\left\|x_n - q\right\| + \left\|x_{n+1} - q\right\|\right) \\ &\qquad + 2\xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n \left\|B_1x_n - B_1q\right\| \|x_n - u_n\| \\ &\qquad + 2\xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda_1 \left\|u_n - y_n\right\| \left\|A_1u_n - A_1q\right\| \\ &\qquad + 2\xi_n\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \left\|\gamma f(x_n) - Aq\right\| \left\|y_n - q\right\| \\ &\qquad + 2(1-\xi_n)s_n \left\|B_2x_n - B_2q\right\| \|x_n - v_n\| + 2(1-\xi_n)\lambda_2 \|v_n - w_n\| \left\|A_2v_n - A_2q\right\|.\end{aligned}$$

By conditions (C1)-(C4), $\lim_{n\to\infty} ||x_n\text{-}x_{n+1}|| = 0$, $\lim_{n\to\infty} ||B_1x_n\text{-}B_1q|| = 0$, $\lim_{n\to\infty} ||B_2x_n\text{-}B_2q|| = 0$, $\lim_{n\to\infty} ||A_1u_n - A_1q|| = 0$ and $\lim_{n\to\infty} ||A_2v_n - A_2q|| = 0$. So, we have $||x_n - u_n|| \to 0$, $||u_n - y_n|| \to 0$, $||x_n - v_n|| \to 0$, $||v_n - w_n|| \to 0$ as $n \to \infty$.

From (2.1), we have

$$\begin{aligned}
\|y'_n - q\|^2 &= \|P_C J_{M_1\lambda_1}(u_n - \lambda_1 A_1 u_n) - P_C J_{M_1\lambda_1}(q - \lambda_1 A_1 q)\|^2 \\
&\le \|J_{M_1,\lambda_1}(u_n - \lambda_1 A_1 u_n) - J_{M_1,\lambda_1}(q - \lambda_1 A_1 q)\|^2 \\
&\le \langle (u_n - \lambda_1 A_1 u_n) - (q - \lambda_1 A_1 q), y'_n - q\rangle \\
&= \frac{1}{2}(\|(u_n - \lambda_1 A_1 u_n) - (q - \lambda_1 A_1 q)\|^2 + \|y'_n - q\|^2 \\
&\quad - \|(u_n - \lambda_1 A_1 u_n) - (q - \lambda_1 A_1 q) - (y'_n - q)\|^2) \\
&= \frac{1}{2}(\|u_n - q\|^2 + \|y'_n - q\|^2 - \|(u_n - y'_n) - \lambda_1(A_1 u_n - A_1 q)\|^2) \\
&\le \frac{1}{2}(\|u_n - q\|^2 + \|y'_n - q\|^2 - \|u_n - y'_n\|^2 \\
&\quad + 2\lambda_1 \langle u_n - y'_n, A_1 u_n - A_1 q\rangle - \lambda_1^2 \|A_1 u_n - A_1 q\|^2),
\end{aligned}$$

which implies that

$$\|y'_n - q\|^2 \le \|u_n - q\|^2 - \|u_n - y'_n\|^2 + 2\lambda_1 \|u_n - y'_n\| \|A_1 u_n - A_1 q\|. \tag{3.37}$$

Substitute (3.30) into (3.37), we have

$$\begin{aligned}
\|y'_n - q\|^2 \le \{&\|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r_n \|B_1 x_n - B_1 q\| \|x_n - u_n\|\} \\
&- \|u_n - y'_n\|^2 + 2\lambda_1 \|u_n - y'_n\| \|A_1 u_n - A_1 q\|.
\end{aligned} \tag{3.38}$$

We consider this inequality in (3.24) that

$$\begin{aligned}
\|z_n - q\|^2 \le{}& \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y'_n - q\|^2 \\
&+ 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\| \|y'_n - q\|.
\end{aligned} \tag{3.39}$$

Substitute (3.38) into (3.39), we have

$$\begin{aligned}
\|z_n - q\|^2 \le{}& \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\{\|x_n - q\|^2 - \|x_n - u_n\|^2 \\
&+ 2r_n \|B_1 x_n - B_1 q\| \|x_n - u_n\| - \|u_n - y'_n\|^2 + 2\lambda_1 \|u_n - y'_n\| \|A_1 u_n - A_1 q\|\} \\
&+ 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\| \|y'_n - q\| \\
\le{}& \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2 \\
&+ 2\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n \|B_1 x_n - B_1 q\| \|x_n - u_n\| - \|u_n - y'_n\|^2 \\
&+ 2\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1 \|u_n - y'_n\| \|A_1 u_n - A_1 q\| \\
&+ 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\| \|y'_n - q\|.
\end{aligned} \tag{3.40}$$

Substitute (3.36) and (3.40) into (3.25), we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \xi_n\|z_n-q\|^2+(1-\xi_n)\|w_n-q\|^2\\
&\le \xi_n\Big\{\alpha_n\|\gamma f(x_n)-Aq\|^2+\|x_n-q\|^2-\|x_n-u_n\|^2-\|u_n-y'_n\|^2\\
&\quad+2\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n-B_1q\|\;\|x_n-u_n\|\\
&\quad+2\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1\|u_n-y'_n\|\;\|A_1u_n-A_1q\|\\
&\quad+2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\;\|y'_n-q\|\Big\}\\
&\quad+(1-\xi_n)\{\|x_n-q\|^2-\|x_n-v_n\|^2+2s_n\|B_2x_n-B_2q\|\;\|x_n-v_n\|-\|v_n-w_n\|^2\\
&\quad+2\lambda_2\|v_n-w_n\|\;\|A_2v_n-A_2q\|\}\\
&\le \xi_n\alpha_n\|\gamma f(x_n)-Aq\|^2+\xi_n\|x_n-q\|^2-\|x_n-u_n\|^2-\|u_n-y'_n\|^2\\
&\quad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n-B_1q\|\;\|x_n-u_n\|\\
&\quad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1\|u_n-y'_n\|\;\|A_1u_n-A_1q\|\\
&\quad+2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\;\|y'_n-q\|\\
&\quad+(1-\xi_n)\|x_n-q\|^2-\|x_n-v_n\|^2+2(1-\xi_n)s_n\|B_2x_n-B_2q\|\;\|x_n-v_n\|\\
&\quad-\|v_n-w_n\|^2+2(1-\xi_n)\lambda_2\|v_n-w_n\|\;\|A_2v_n-A_2q\|.
\end{aligned}$$

Then, we derive

$$\begin{aligned}
&\|x_n-u_n\|^2+\|u_n-y'_n\|^2+\|x_n-v_n\|^2+\|v_n-w_n\|^2\\
&\le \xi_n\alpha_n\|\gamma f(x_n)-Aq\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2\\
&\quad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n-B_1q\|\;\|x_n-u_n\|\\
&\quad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1\|u_n-y'_n\|\;\|A_1u_n-A_1q\|\\
&\quad+2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\;\|y'_n-q\|\\
&\quad+2(1-\xi_n)s_n\|B_2x_n-B_2q\|\;\|x_n-v_n\|+2(1-\xi_n)\lambda_2\|v_n-w_n\|\;\|A_2v_n-A_2q\|\\
&= \xi_n\alpha_n\|\lambda f(x_n)-Aq\|^2+\|x_{n+1}-x_n\|\left(\|x_n-q\|+\|x_{n+1}-q\|\right)\\
&\quad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n-B_1q\|\;\|x_n-u_n\|\\
&\quad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda_1\|u_n-y'_n\|\;\|A_1u_n-A_1q\|\\
&\quad+2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n)-Aq\|\;\|y'_n-q\|\\
&\quad+2(1-\xi_n)s_n\|B_2x_n-B_2q\|\;\|x_n-v_n\|+2(1-\xi_n)\lambda_2\|v_n-w_n\|\;\|A_2v_n-A_2q\|.
\end{aligned}$$

By conditions (C1)-(C4), $\lim_{n\to\infty} ||x_n\text{-}x_{n+1}|| = 0$, $\lim_{n\to\infty} ||B_1x_n\text{-}B_1q|| = 0$, $\lim_{n\to\infty} ||B_2x_n\text{-}B_2q|| = 0$, $\lim_{n\to\infty} ||A_1u_n - A_1q|| = 0$, $\lim_{n\to\infty} ||A_2v_n - A_2q|| = 0$, $||x_n - u_n|| \to 0$, $||x_n - v_n|| \to 0$, $||v_n - w_n|| \to 0$ as $n \to \infty$. So, we have $\|u_n - y'_n\| \to 0$. It follows that

$$\|x_n - w_n\| \le \|x_n - v_n\| + \|v_n - w_n\| \to 0, \quad \text{as } n \to \infty.$$

We compute that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \left\|\xi_n(z_n - x_n) + (1 - \xi_n)(P_C w_n - x_n)\right\| \\ &\le \xi_n \|z_n - x_n\| + (1 - \xi_n)\|w_n - x_n\|.\end{aligned}$$

From $\lim_{n\to\infty} ||x_n - w_n|| = 0$, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, and hence

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.41}$$

It follows by step 4 (i) and (ii),

$$\left\|x_n - y_n\right\| \le \|x_n - u_n\| + \left\|u_n - y_n\right\| \to 0, \quad \text{as } n \to \infty.$$

Since

$$\left\|z_n - y_n\right\| \le \|z_n - x_n\| + \left\|x_n - y_n\right\|.$$

So, by (3.41) and $\lim_{n\to\infty} ||x_n - y_n|| = 0$, we obtain

$$\lim_{n\to\infty} \left\|z_n - y_n\right\| = 0. \tag{3.42}$$

We show $\left\|Sy'_n - z_n\right\| \to 0$ as $n \to \infty$. By nonexpansiveness of P_C notice that

$$\begin{aligned}\left\|Sy'_n - z_n\right\| &= \left\|Sy'_n - P_C[\alpha_n\gamma f(x_n) + (I - \alpha_n A)Sy'_n]\right\| \\ &= \left\|P_C Sy'_n - P_C[\alpha_n\gamma f(x_n) + (I - \alpha_n A)Sy'_n]\right\| \\ &\le \left\|Sy'_n - \alpha_n\gamma f(x_n) - (I - \alpha_n A)Sy'_n\right\| \\ &\le \alpha_n \left\|\gamma f(x_n) - ASy'_n\right\|.\end{aligned}$$

By condition (C1), we get $\lim_{n\to\infty} \left\|Sy'_n - z_n\right\| = 0$. Since $\left\|Sy'_n - y'_n\right\| \le \left\|Sy'_n - z_n\right\| + \left\|z_n - y_n\right\| + \left\|y_n - u_n\right\| + \left\|u_n - y'_n\right\|$, so by (3.42), $\lim_{n\to\infty} \left\|Sy'_n - z_n\right\| = 0, \lim_{n\to\infty} \left\|y_n - u_n\right\| = 0$ and $\lim_{n\to\infty} \left\|u_n - y'_n\right\| = 0$, we obtain $\lim_{n\to\infty} \left\|Sy'_n - y'_n\right\| = 0$.

Step 5. We show that $q \in \Theta := F(S) \cap \text{GMEP}(F_1, \phi_1, B_1) \cap \text{GMEP}(F_2, \phi_2, B_2) \cap I(A_1, M_1) \cap I(A_2, M_2)$ and $\limsup_{n\to\infty} \left\langle(\gamma f - A)q, Sy'_n - q\right\rangle \le 0$. It is easy to see that $P_\Theta(\gamma f + (I - A))$ is a contraction of H into itself. In fact, from Lemma 2.9, we have

$$\begin{aligned}\left\|P_\Theta(\gamma f + (I - A))x - P_\Theta(\gamma f + (I - A))y\right\| &\le \left\|(\gamma f + (I - A))x - (\gamma f + (I - A))y\right\| \\ &\le \gamma \left\|f(x) - f(y)\right\| + (I - A)\left\|x - y\right\| \\ &\le \gamma\alpha \left\|x - y\right\| + \left(1 - \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)\right)\left\|x - y\right\| \\ &= \left(\sqrt{\frac{1 - \delta}{\mu}} + \gamma\alpha\right)\left\|x - y\right\|.\end{aligned}$$

Hence H is complete, there exists a unique fixed point $q \in H$ such that $q = P_\Theta(\gamma f + (I - A))(q)$. By Lemma 2.2 we obtain that $\langle(\gamma f - A)q, w - q\rangle \le 0$ for all $w \in \Theta$.

Next, we show that $\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy'_n - q\rangle \le 0$, where $q = P_\Theta(\gamma f + I - A)(q)$ is the unique solution of the variational inequality $\langle(\gamma f - A)q, p - q\rangle \ge 0, \forall p \in \Theta$. We can choose a subsequence $\{y'_{n_i}\}$ of $\{y'_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy'_n - q\rangle = \lim_{i\to\infty}\langle(\gamma f - A)q, Sy'_{n_i} - q\rangle.$$

Since $\{y'_{n_i}\}$ is bounded, there exists a subsequence $\{y'_{n_{i_j}}\}$ of $\{y'_{n_i}\}$ which converges weakly to w. We may assume without loss of generality that $y'_{n_i} \rightharpoonup w$.

We claim that $w \in \Theta$. Since $\|y'_n - Sy'_n\| \to 0$ and by Lemma 2.6, we have $w \in F(S)$.

Next, we show that $w \in \mathrm{GMEP}(F_1, \phi_1, B_1)$. Since $u_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n B_1 x_n)$, we know that

$$F_1(u_n, y) + \varphi_1(y) - \varphi_1(u_n) + \langle B_1 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

It follows by (A2) that

$$\varphi_1(y) - \varphi_1(u_n) + \langle B_1 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F_1(y, u_n), \quad \forall y \in C.$$

Hence,

$$\varphi_1(y) - \varphi_1(u_{n_i}) + \langle B_1 x_{n_i}, y - u_{n_i}\rangle + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \ge F_1(y, u_{n_i}), \quad \forall y \in C. \tag{3.43}$$

For $t \in (0, 1]$ and $y \in H$, let $y_t = ty + (1 - t)w$. From (3.43), we have

$$\begin{aligned}\langle y_t - u_{n_i}, B_1 y_t\rangle \ge{}& \langle y_t - u_{n_i}, B_1 y_t\rangle - \varphi_1(y_t) + \varphi_1(u_{n_i}) - \langle B_1 x_{n_i}, y_t - u_{n_i}\rangle \\ & - \frac{1}{r_{n_i}}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F_1(y_t, u_{n_i}) \\ ={}& \langle y_t - u_{n_i}, B_1 y_t - B_1 u_{n_i}\rangle + \langle y_t - u_{n_i}, B_1 u_{n_i} - B_1 x_{n_i}\rangle - \varphi_1(y_t) + \varphi_1(u_{n_i}) \\ & - \frac{1}{r_{n_i}}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F_1(y_t, u_{n_i}).\end{aligned}$$

From $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|B_1 u_{n_i} - B_1 x_{n_i}\| \to 0$. Further, from (A4) and the weakly lower semicontinuity of φ_1, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, we have

$$\langle y_t - w, B_1 y_t\rangle \ge -\varphi_1(y_t) + \varphi_1(w) + F_1(y_t, w). \tag{3.44}$$

From (A1), (A4) and (3.44), we have

$$\begin{aligned}
0 &= F_1(y_t, y_t) - \varphi_1(y_t) + \varphi_1(y_t)\\
&\le tF_1(y_t, y) + (1-t)F_1(y_t, w) + t\varphi_1(y) + (1-t)\varphi_1(w) - \varphi_1(y_t)\\
&= t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)[F_1(y_t, w) + \varphi_1(w) - \varphi_1(y_t)]\\
&\le t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)\langle y_t - w, B_1 y_t\rangle\\
&= t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)t\langle y - w, B_1 y_t\rangle,
\end{aligned}$$

and hence

$$0 \le F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t) + (1-t)\langle y - w, B_1 y_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F_1(w, y) + \varphi_1(y) - \varphi_1(w) + \langle y - w, B_1 w\rangle \ge 0.$$

This implies that $w \in$ GMEP(F_1, ϕ_1, B_1). By following the same arguments, we can show that $w \in$ GMEP(F_2, ϕ_2, B_2).

Lastly, we show that $w \in I(A_1, M_2)$. In fact, since A_1 is a β_1-inverse-strongly monotone, A_1 is monotone and Lipschitz continuous mapping. It follows from Lemma 2.3 that $M_1 + A_1$ is a maximal monotone. Let $(v, g) \in G(M_1 + A_1)$, since $g - A_1 v \in M_1(v)$. Again since $y_{n_i} = J_{M_1\lambda_1}(u_{n_i} - \lambda_1 A_1 u_{n_i})$, we have $u_{n_i} - \lambda_1 A_1 u_{n_i} \in (I + \lambda_1 M_1)(y_{n_i})$, that is, $\frac{1}{\lambda_1}(u_{n_i} - y_{n_i} - \lambda_1 A_1 u_{n_i}) \in M_1(y_{n_i})$. By virtue of the maximal monotonicity of M_1 + A_1, we have

$$\left\langle v - y_{n_i}, g - A_1 v - \frac{1}{\lambda_1}(u_{n_i} - y_{n_i} - \lambda_1 A_1 u_{n_i})\right\rangle \ge 0,$$

and hence

$$\begin{aligned}
\langle v - y_{n_i}, g\rangle &\ge \left\langle v - y_{n_i}, A_1 v + \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda_1 A_1 u_{n_i})\right\rangle\\
&= \langle v - y_{n_i}, A_1 v - A_1 y_{n_i}\rangle + \langle v - y_{n_i}, A_1 y_{n_i} - A_1 u_{n_i}\rangle\\
&\quad + \left\langle v - y_{n_i}, \frac{1}{\lambda_1}(u_{n_i} - y_{n_i})\right\rangle.
\end{aligned}$$

It follows from $\lim_{n\to\infty} ||u_n - y_n|| = 0$, we have $\lim_{n\to\infty} ||A_1 u_n - A_1 y_n|| = 0$ and $y_{n_i} \rightharpoonup w$ that

$$\limsup_{n\to\infty} \langle v - y_n, g\rangle = \langle v - w, g\rangle \ge 0.$$

It follows from the maximal monotonicity of $A_1 + M_1$ that $\theta \in (M_1 + A_1)(w)$, that is, $w \in I(A_1, M_1)$. By following the same arguments, we can show that $w \in I(A_2, M_2)$. Therefore, $w \in \Theta$. It follows that

$$\limsup_{n\to\infty} \langle (\gamma f - A)q, Sy'_n - q\rangle = \lim_{i\to\infty} \langle (\gamma f - A)q, Sy'_{n_i} - q\rangle = \langle (\gamma f - A)q, w - q\rangle \le 0.$$

Step 6. We prove $x_n \to q$. By using (3.1), Lemma 2.9 (ii) and together with Schwarz inequality, we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\xi_n P_C[(\alpha_n \gamma f(x_n) + (I - \alpha_n A)Sy'_n) - q] + (1 - \xi_n)(P_C w_n - q)\|^2 \\
&\leq \xi_n \|P_C[(\alpha_n \gamma f(x_n) + (I - \alpha_n A)Sy'_n) - P_C(q)]\|^2 + (1 - \xi_n)\|w_n - q\|^2 \\
&\leq \xi_n \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(Sy'_n - q)\|^2 + (1 - \xi_n)\|x_n - q\|^2 \\
&\leq \xi_n (I - \alpha_n A)^2 \|Sy'_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n \alpha_n \langle (I - \alpha_n A)(Sy'_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n)\|x_n - q\|^2 \\
&\leq \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)^2 \|y_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n \alpha_n \langle Sy'_n - q, \gamma f(x_n) - Aq \rangle \\
&\quad - 2\xi_n \alpha_n^2 \langle A(Sy'_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n)\|x_n - q\|^2 \\
&\leq \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)^2 \|x_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n \alpha_n \langle Sy'_n - q, \gamma f(x_n) - \gamma f(q) \rangle + 2\xi_n \alpha_n \langle Sy'_n - q, \gamma f(q) - Aq \rangle \\
&\quad - 2\xi_n \alpha_n^2 \langle A(Sy'_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n)\|x_n - q\|^2 \\
&\leq \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)^2 \|x_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n \alpha_n \|Sy'_n - q\| \|\gamma f(x_n) - \gamma f(q)\| + 2\xi_n \alpha_n \langle Sy'_n - q, \gamma f(q) - Aq \rangle \\
&\quad - 2\xi_n \alpha_n^2 \langle A(Sy'_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n)\|x_n - q\|^2 \\
&\leq \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)^2 \|x_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n \gamma \alpha \alpha_n \|y_n - q\| \|x_n - q\| + 2\xi_n \alpha_n \langle Sy'_n - q, \gamma f(q) - Aq \rangle \\
&\quad - 2\xi_n \alpha_n^2 \langle A(y'_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n)\|x_n - q\|^2 \\
&\leq \left(\xi_n - 2\xi_n \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) + \xi_n \alpha_n^2 \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)^2\right) \|x_n - q\|^2 \\
&\quad + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 + 2\xi_n \gamma \alpha \alpha_n \|x_n - q\|^2 \\
&\quad + 2\xi_n \alpha_n \langle Sy'_n - q, \gamma f(q) - Aq \rangle \\
&\quad - 2\xi_n \alpha_n^2 \langle A(Sy'_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n)\|x_n - q\|^2 \\
&\leq \left(1 - 2\xi_n \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) + 2\xi_n \gamma \alpha \alpha_n\right) \|x_n - q\|^2 + \alpha_n \Bigg\{\xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n \langle Sy'_n - q, \gamma f(q) - Aq \rangle - 2\xi_n \alpha_n \|A(Sy'_n - q)\| \|\gamma f(x_n) - Aq\| \\
&\quad + \xi_n \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)^2 \|x_n - q\|^2 \Bigg\} \\
&= \left(1 - 2\left(\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) - \gamma \alpha\right) \xi_n \alpha_n\right) \|x_n - q\|^2 + \alpha_n \Bigg\{\xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\xi_n \langle Sy'_n - q, \gamma f(q) - Aq \rangle - 2\xi_n \alpha_n \|A(Sy'_n - q)\| \|\gamma f(x_n) - Aq\| \\
&\quad + \xi_n \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)^2 \|x_n - q\|^2 \Bigg\}.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, where $\eta \geq \xi_n \|\gamma f(x_n) - Aq\|^2 - 2\xi_n \|A(Sy'_n - q)\| \|\gamma f(x_n) - Aq\| + \xi_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)^2 \|x_n - q\|^2$ for all $n \geq 0$. It follows that

$$\|x_{n+1} - q\|^2 \leq \left(1 - 2\left(\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) - \gamma\alpha\right)\xi_n\alpha_n\right)\|x_n - q\|^2\alpha_n\varsigma_n,$$

where $\varsigma_n = 2\xi_n \langle Sy'_n - q, \gamma f(q) - Aq\rangle + \eta\alpha_n$. By $\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy'_n - q\rangle \leq 0$, we get $\limsup_{n\to\infty}\varsigma_n \leq 0$. Applying Lemma 2.5, we can conclude that $x_n \to q$. This completes the proof. □

Next, the following example shows that all conditions of Theorem 3.1 are satisfied.

Example 3.2. For instance, let $\alpha_n = \frac{1}{2(n+1)}, \xi_n = \frac{2n+2}{2(2n)}, r_n = \frac{n}{n+1}$ and $s_n = \frac{n}{n+1}$, then, we will show that the sequences $\{\alpha_n\}$ satisfy the condition (C1). Indeed, we take $\alpha_n = \frac{1}{2(n+1)}$ then we have

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{1}{2(n+1)} = 0$$

$$\sum_{n=1}^{\infty}\alpha_n = \sum_{n=1}^{\infty}\frac{1}{2(n+1)} = \infty$$

and

$$\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| = \sum_{n=1}^{\infty}\left|\frac{1}{2(n+1)} - \frac{1}{2n}\right| \leq \left|\frac{1}{2.2} - \frac{1}{2.1}\right| + \left|\frac{1}{2.3} - \frac{1}{2.2}\right| + \left|\frac{1}{2.4} - \frac{1}{2.3}\right| + \cdots = \frac{1}{2}.$$

Then, the sequence $\{\alpha_n\}$ satisfy the condition (C1).

We will show that the sequences $\{\xi_n\}$ satisfy the condition (C2). Indeed, we set $\xi_n = \frac{2n+2}{2(2n)} = \frac{1}{2} + \frac{1}{2n}$. By similarly (C1), it is easy to see that $0 < \liminf_{n\to\infty}\xi_n < \limsup_{n\to\infty}\xi_n < 1$ and $\sum_{n=1}^{\infty}|\xi_{n+1} - \xi_n| < \infty$.

Next, we will show that the condition (C3) is achieves. We take $r_n = \frac{n}{n+1}$, then we compute

$$\sum_{n=1}^{\infty}|r_n - r_{n-1}| = \sum_{n=1}^{\infty}\left|\frac{n}{n+1} - \frac{n-1}{(n-1)+1}\right| = \sum_{n=1}^{\infty}\left|\frac{n(n) - (n-1)(n+1)}{(n+1)n}\right|$$
$$= \sum_{n=1}^{\infty}\left|\frac{n^2 - n^2 + 1}{(n+1)n}\right| = \sum_{n=1}^{\infty}\left|\frac{1}{n(n+1)}\right|.$$

Then, we have $\liminf_{n\to\infty} r_n = \liminf_{n\to\infty}\frac{n}{n+1} = 1$ and also $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

The sequence $\{r_n\}$ satisfy the condition (C3). In the same way with (C4). □

Corollary 3.3. *Let H be a real Hilbert space, C be a closed convex subset of H. Let F_1, F_2 be bifunctions of $C \times C$ into $\mathcal{R}$ satisfying* (A1)-(A4) *and $B, B_1, B_2 : C \to H$ be β, η, ρ-inverse-strongly monotone mappings, $\varphi_1, \varphi_2 : C \to \mathcal{R}$ be convex and lower semicontinuous functions, $f : C \to C$ be a contraction with coefficient α* $(0 < \alpha < 1)$, *$M : H \to 2^H$ be a maximal monotone mapping. Assume that either* (B1) *or* (B2) *holds. Let S be a nonexpansive mapping of C into itself such that*

$$\Theta_1 := F(S) \cap \mathrm{GMEP}(F_1, \varphi_1, B_1) \cap \mathrm{GMEP}(F_2, \varphi_2, B_2) \cap I(B, M) \neq \emptyset.$$

Suppose $\{x_n\}$ *is a sequences generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n B_1 x_n) \\ v_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B_2 x_n) \\ x_{n+1} = \xi_n P_C[\alpha_n f(x_n) + (I - \alpha_n) S P_C(J_{M,\lambda}(I - \lambda B) u_n)] + (1 - \xi_n) P_C v_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\xi_n\} \subset (0, 1)$, $\lambda \in (0, 2\beta)$ *such that* $0 < a \le \lambda \le b < 2\beta$, $r_n \in (0, 2\eta)$ *with* $0 < c \le d \le 1 - \eta$ *and* $s_n \in (0, 2\rho)$ *with* $0 < e \le f \le 1 - \rho$ *satisfy the following conditions:*

(C1): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C2): $0 < \liminf_{n\to\infty} \xi_n < \limsup_{n\to\infty} \xi_n < 1$, $\sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty$,

(C3): $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(C4): $\liminf_{n\to\infty} s_n > 0$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta_1$, *where* $q = P_{\Theta_1}(f + I)(q)$ *which solves the following variational inequality:*

$$\langle (f - I)q, p - q \rangle \le 0, \quad \forall p \in \Theta_1.$$

Proof. Putting $A \equiv I$, $\gamma \equiv 1$, $J_{M_2,\lambda_2} \equiv I$ and $A_2 \equiv 0$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Corollary 3.4. *Let H be a real Hilbert space, C be a closed convex subset of H. Let* F_1, F_2 *be bifunctions of* $C \times C$ *into* $\mathcal{R}$ *satisfying (A1)-(A4) and* $B, B_1, B_2 : C \to H$ *be* β, η, ρ-*inverse-strongly monotone mappings,* $\varphi_1, \varphi_2 : C \to \mathcal{R}$ *be convex and lower semicontinuous functions. Assume that either (B1)or (B2) holds. Let S be a nonexpansive mapping of C into itself such that*

$$\Theta_1 := F(S) \cap \mathrm{GMEP}(F_1, \varphi_1, B_1) \cap \mathrm{GMEP}(F_2, \varphi_2, B_2) \cap I(B, M) \neq \emptyset.$$

Suppose $\{x_n\}$ *is a sequences generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n B_1 x_n) \\ v_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B_2 x_n) \\ x_{n+1} = \xi_n P_C[\alpha_n u + (I - \alpha_n) S P_C(J_{M,\lambda}(I - \lambda B) u_n)] + (1 - \xi_n) P_C v_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\xi_n\} \subset (0, 1)$, $\lambda \in (0, 2\beta)$ *such that* $0 < a \le \lambda \le b < 2\beta$, $r_n \in (0, 2\eta)$ *with* $0 < c \le d \le 1 - \eta$ *and* $s_n \in (0, 2\rho)$ *with* $0 < e \le f \le 1 - \rho$ *satisfy the following conditions:*

(C1): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C2): $0 < \liminf_{n\to\infty} \xi_n < \limsup_{n\to\infty} \xi_n < 1$, $\sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty$,

(C3): $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(C4): $\liminf_{n\to\infty} s_n > 0$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta_1$, *where* $q = P_{\Theta_1}(q)$ *which solves the following variational inequality:*

$$\langle u - q, p - q \rangle \le 0, \quad \forall p \in \Theta_1.$$

Proof. Putting $f \equiv u \in C$ in Corollary 3.3, we can obtain desired conclusion immediately. □

Corollary 3.5. *Let H be a real Hilbert space, C be a closed convex subset of H. Let* F_1, F_2 *be bifunctions of* $C \times C$ *into* $\mathcal{R}$ *satisfying* (A1)-(A4) *and* $B, B_1, B_2 : C \to H$ *be* β, η, ρ-*inverse-strongly monotone mappings,* $\varphi_1, \varphi_2 : C \to \mathcal{R}$ *be convex and lower semicontinuous functions,* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$ *and A is a* δ-*strongly monotone and* μ-*strictly pseudo-contraction with* $\delta + \mu > 1$, γ *is a positive real number such that* $\gamma < \frac{1}{\alpha}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)$. *Assume that either* (B1) *or* (B2) *holds. Let S be a nonexpansive mapping of C into itself such that*

$$\Theta_2 := F(S) \cap \mathrm{GMEP}(F_1, \varphi_1, B_1) \cap \mathrm{GMEP}(F_2, \varphi_2, B_2) \cap VI(C, B) \neq \emptyset.$$

Suppose $\{x_n\}$ *is a sequences generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n B_1 x_n) \\ v_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B_2 x_n) \\ x_{n+1} = \xi_n P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) S P_C (I - \lambda B) u_n] + (1 - \xi_n) P_C v_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\xi_n\} \subset (0, 1)$, $\lambda \subset (0, 2\beta)$ *such that* $0 < a \leq \lambda \leq b < 2\beta$, $r_n \in (0, 2\eta)$ *with* $0 < c \leq d \leq 1 - \eta$ *and* $s_n \in (0, 2\rho)$ *with* $0 < e \leq f \leq 1 - \rho$ *satisfy the following conditions:*

(C1): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C2): $0 < \liminf_{n\to\infty} \xi_n < \limsup_{n\to\infty} \xi_n < 1$, $\sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty$,

(C3): $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(C4): $\liminf_{n\to\infty} s_n > 0$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta_2$, *where* $q = P_{\Theta_2}(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle (\gamma f - A)q, p - q \rangle \leq 0, \quad \forall p \in \Theta_2.$$

Proof. Taking $J_{M_1,\lambda_1} \equiv I, J_{M_2,\lambda_2} \equiv I, A_1 \equiv B$ and $A_2 \equiv 0$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Corollary 3.6. *Let H be a real Hilbert space, C be a closed convex subset of H. Let* F_1, F_2 *be bifunctions of* $C \times C$ *into* $\mathcal{R}$ *satisfying* (A1)-(A4) *and* B_1, $B_2 : C \to H$ *be* η, ρ-*inverse-strongly monotone mappings,* $\varphi_1, \varphi_2 : C \to \mathcal{R}$ *be convex and lower semicontinuous functions,* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$. *Assume that either* (B1) *or* (B2) *holds. Let S be a nonexpansive mapping of C into itself such that*

$$\Theta_3 := F(S) \cap \mathrm{GMEP}(F_1, \varphi_1, B_1) \cap \mathrm{GMEP}(F_2, \varphi_2, B_2) \neq \emptyset.$$

Suppose $\{x_n\}$ *is a sequences generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n B_1 x_n) \\ v_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B_2 x_n) \\ x_{n+1} = \xi_n P_C[\alpha_n f(x_n) + (I - \alpha_n) S P_C u_n] + (1 - \xi_n) P_C v_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\xi_n\} \subset (0, 1)$, $r_n \in (0, 2\eta)$ *with* $0 < c \leq d \leq 1 - \eta$ *and* $s_n \in (0, 2\rho)$ *with* $0 < e \leq f \leq 1 - \rho$ *satisfy the following conditions:*

(C1): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C2): $0 < \liminf_{n\to\infty} \xi_n < \limsup_{n\to\infty} \xi_n < 1$, $\sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty$,

(C3): $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, (C4): $\liminf_{n\to\infty} s_n > 0$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$.

Then $\{x_n\}$ *converges strongly to* $q \in \Theta_3$, *where* $q = P_{\Theta_3}(f + I)(q)$ *which solves the following variational inequality:*

$$\langle (f - I)q, p - q \rangle \leq 0, \quad \forall p \in \Theta_3$$

Proof. Taking $\gamma \equiv 1$, $A \equiv I$ and $B \equiv 0$ in Corollary 3.5, we can obtain desired conclusion immediately. □

4. Acknowledgements
The authors are grateful for the reviewers for the careful reading of the article and for the suggestions which improved the quality of this work.
This research was partially supported by the Hands-On Research and Development, Rajamangala University of Technology Lanna (under Project No. UR2L-003). Furthermore, the third authors also would like to thank the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU-CSEC Project No.55000613) for financial support.

Author details
[1]Department of Mathematics and Statistics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Tak, Tak 63000, Thailand [2]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology (KMUTT), Bangmod, Thrungkru, Bangkok 10140, Thailand

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 6 March 2012 Accepted: 13 June 2012 Published: 13 June 2012

References

1. Browder, FE: Existence and approximation of solutions of nonlinear variational inequalities. Proc Natl Acad Sci USA. **56**, 1080–1086 (1966). doi:10.1073/pnas.56.4.1080
2. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math Student. **63**, 123–145 (1994)
3. Chang, SS, Lee, HWJ, Chan, CK: A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. Nonlinear Anal Theory Method Appl. **670**, 3307–3319 (2009)
4. Shehu, Y: A new iterative scheme for a countable family of relatively nonex- pansive mappings and an equilibrium problem in Banach spaces. J Global Optim
5. Ye, J, Huang, J: Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces. J Math Comput Sci. **1**, 1–18 (2011)
6. Yang, S, Li, W: Iterative solutions of a system of equilibrium problems in Hilbert spaces. Adv Fixed Point Theory. **1**, 15–26 (2011)
7. Hartman, P, Stampacchia, G: On some nonlinear elliptic differential functional equations. Acta Math. **115**, 271–310 (1966). doi:10.1007/BF02392210
8. Marino, G, Xu, HK: A general iterative method for nonexpansive mapping in Hilbert space. J Math Anal Appl. **318**, 43–52 (2006). doi:10.1016/j.jmaa.2005.05.028
9. Kirk, WA: Fixed point theorem for mappings which do not increase distance. Am Math Monthly. **72**, 1004–1006 (1965). doi:10.2307/2313345
10. Browder, FE, Petryshym, WV: Construction of fixed points of nonlinear mappings in Hilbert spaces. J Math Anal Appl. **20**, 197–228 (1963)
11. Liu, M, Chang, SS, Zuo, P: An algorithm for finding a common solution for a system of mixed equilibrium problem, quasivariational inclusion problems of nonexpansive semigroup. J Inequal Appl 23 (2010). Article ID 895907,
12. Hao, Y: Some results of variatinal inclusion problems and fixed point problems with applications. Appl Math Mech. **30**(12), 1589–1596 (2009). doi:10.1007/s10483-009-1210-x
13. Matsushita, SY, Takahashi, W: Existence theorems for set-valued operators in Banach spaces. Set-Valued Anal. **15**, 251–264 (2007). doi:10.1007/s11228-006-0030-8
14. Tan, JF, Chang, SS: Iterative algorithms for finding common solutions to variational inclusion equilibrium and fixed point problems. Fixed Point Theory Appl 17 (2011). Article ID 915629,
15. Zhang, SS, Joseph Lee, HW, Chan, CK: Algorithms of common solutions to quasi variational inclusion and fixed point problems. Appl Math Mech. **29**(5) (2008)
16. Rockafellar, RT: On the maximality of sums of nonlinear monotone operators. Trans Am Math Soc. **149**, 46–55 (2000)
17. Takahashi, W: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
18. Moudafi, A: Viscosity approximation methods for fixed points problems. J Math Anal Appl. **241**, 46–55 (2000). doi:10.1006/jmaa.1999.6615

19. Iiduka, H, Takahashi, W: Strong convergence theorems for nonexpansive mapping and inverse-strong monotone mappings. Nonlinear Anal. **61**, 341–350 (2005). doi:10.1016/j.na.2003.07.023
20. Su, Y, Shang, M, Qin, X: An iterative method of solution for equilibrium and optimization problems. Nonlinear Anal. **69**, 2709–2719 (2008). doi:10.1016/j.na.2007.08.045
21. Brézis, H: Opérateur maximaux monotones. In Mathematics Studies, vol. 5,North-Holland, Amsterdam (1973)
22. Opial, Z: Weak convergence of successive approximations for nonexpansive mappings. Bull Am Math Soc. **73**, 591–597 (1967). doi:10.1090/S0002-9904-1967-11761-0
23. Xu, HK: Iterative algorithms for nonlinear operators. J Lond Math Soc. **66**, 240–256 (2002). doi:10.1112/S0024610702003332
24. Browder, FE: Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proc Symp Pure Math. **18**, 78–81 (1976)
25. Peng, J-W, Liou, Y-C, Yao, J-C: An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions. Fixed Point Theory Appl 21 (2009). Article ID 794178,
26. Wangkeeree, R, Bantaojai, T: A general composite algorithms for solving general equilibrium problems and fixed point problems in Hilbert spaces. Thai J Math. **9**, 191–212 (2011)

doi:10.1186/1029-242X-2012-138
Cite this article as: Wattanawitoon *et al.*: **A new hybrid general iterative algorithm for common solutions of generalized mixed equilibrium problems and variational inclusions.** *Journal of Inequalities and Applications* 2012 **2012**:138.